JOURNAL OF APPROXIMATION THEORY 77, 331–348 (1994)

On L_p-Boundedness of the L_2-Projector onto Splines

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Communicated by Carl de Boor

Received November 19, 1991; accepted in revised form January 29, 1993

In this paper we offer a new approach to C. de Boor's conjecture of the L_∞-boundedness of the L_2-projector P_S onto the spline space $S_{m-1}(\Delta_n)$. This approach is based on the strengthening of the "exponential decay" property of the fundamental spline. It is proved, first, that the L_p-norm of the operator P_S is uniformly bounded without any restrictions on the mesh Δ_n at least in some neighbourhood of $p=2$ and, second, that the L_p-norm of the operator P_S for all $p \in [1, \infty]$ is uniformly bounded in meshes Δ_n with a fixed number of nodes n.

1. INTRODUCTION

For a given partition of the interval $[a, b]$

$$\Delta_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

denote by $S_{m-1}(\Delta_n)$ the space of splines of degree $m-1$ with deficiency 1 on the mesh Δ_n and consider the operator P_S of orthogonal projection onto $S_{m-1}(\Delta_n)$, defined by

$$\int_a^b [f(t) - P_S(f, t)]\, \sigma(t)\, dt = 0 \qquad \forall \sigma \in S_{m-1}(\Delta_n).$$

We are interested in the norm of P_S as an operator from L_p on L_p, i.e., in the quantity

$$l_{m-1}(\Delta_n)_p = \sup_{\|f\|_p \leq 1} \|P_{S_{m-1}(\Delta_n)}(f)\|_p, \qquad p \in [1, \infty]. \tag{1}$$

In the study of this quantity the main guide line is given by the following

0021-9045/94 $6.00

Conjecture [3]. For each $m \in \mathbb{N}$ there is a constant c_m such that for all $n \in \mathbb{N}$ and $\Delta_n \subset [a, b]$

$$l_{m-1}(\Delta_n)_p \leqslant c_m, \qquad p \in [1, \infty].$$

This conjecture is valid for $m = 1, 2, 3$ (see [4]). For $m \geqslant 4$, all known estimates of (1) depend on either parameters of the mesh Δ_n. The most improved one which is also due to C. de Boor, looks as follows. Set

$$\kappa_i := t_{i+m} - t_i, \qquad 1/p' := 1 - 1/p, \quad z_+ := \max(0, z).$$

THEOREM [5]. *For any $m \in \mathbb{N}$ and arbitrary mesh $\Delta_n \subset [a, b]$*

$$l_{m-1}(\Delta_n)_p \leqslant c_m \max_{i,j} (\kappa_i \kappa_j^{-1})^{\theta_0}, \qquad p \in [1, \infty],$$

where

$$\theta_0 := (1/2 - 1/p)_+ + (1/2 - 1/p')_+.$$

The main result of this paper is

THEOREM 1. *For any $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that for each $0 \leqslant \varepsilon < \varepsilon_m$ and arbitrary mesh $\Delta_n \subset [a,b]$*

$$l_{m-1}(\Delta_n)_p \leqslant c_{m,\varepsilon} \max_{i,j} (\kappa_i \kappa_j^{-1})^{\theta}, \qquad p \in [1, \infty],$$

where

$$\theta := \theta_\varepsilon := (1/2 - \varepsilon - 1/p)_+ + (1/2 - \varepsilon - 1/p')_+;$$

in particular for each $0 \leqslant \varepsilon < \varepsilon_m$, uniformly in $n \in \mathbb{N}$ and $\Delta_n \subset [a, b]$,

$$l_{m-1}(\Delta_n)_p \leqslant c_{m,\varepsilon}, \qquad p \in \left[2 - \frac{4\varepsilon}{1+2\varepsilon}, 2 + \frac{4\varepsilon}{1-2\varepsilon}\right].$$

In addition we prove

THEOREM 2. *For any $m \in \mathbb{N}$ there exist $\zeta_m, c_m < \infty$ such that for any $n \in \mathbb{N}$, uniformly in $\Delta_n \subset [a, b]$,*

$$l_{m-1}(\Delta_n)_p \leqslant c_m (\zeta_m)^n, \qquad p \in [1, \infty].$$

Thus, it follows that, first of all, the L-norm of the operator P_S of orthogonal spline projection is *uniformly bounded without any restrictions on the mesh Δ_n in some neighbourhood of $p = 2$ and,* second, the L_p-norm of the operator P_S *for all $p \in [1, \infty]$ is uniformly bounded in meshes Δ_n with a fixed number of nodes n.*

2. Proof of Theorem 1

In Section 3 we construct an orthonormal basis $\varphi = \{\varphi_i\}_{i=1}^{n+m-1}$ for the space $S_{m-1}(\Delta_n)$, and in Sections 4–6 we show that its elements satisfy the following exponential estimate for decay of L_p-norms taken over the subintervals of the mesh Δ_n.

Set

$$\kappa_{\max} := \max_i \kappa_i, \qquad \kappa_{\min} := \min_i \kappa_i, \quad \sigma_m := \kappa_{\max} \kappa_{\min}^{-1}.$$

Lemma 1. *For any $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that for any $0 \leqslant \varepsilon < \varepsilon_m$ there exists $\lambda = \lambda_\varepsilon < 1$, for which for arbitrary mesh $\Delta_n \subset [a, b]$ for all amissible i, j,*

$$\|\varphi_i\|_{L_p[t_j, t_{j+1}]} \leqslant c_m \lambda_\varepsilon^{|i-j|} \|\varphi_i\|_{L_2[0,1]} \times \begin{cases} \kappa_{i-m}^{-\varepsilon} \kappa_{\min}^{-\theta}, & 0 \leqslant \dfrac{1}{p} < \dfrac{1}{2} - \varepsilon; \\ \kappa_{i-m}^{1/p - 1/2}, & \dfrac{1}{2} - \varepsilon \leqslant \dfrac{1}{p} \leqslant \dfrac{1}{2} + \varepsilon; \\ \kappa_{i-m}^{\varepsilon} \kappa_{\max}^{\theta}, & \dfrac{1}{2} + \varepsilon < \dfrac{1}{p} \leqslant 1. \end{cases} \tag{2}$$

Assuming this estimate proved, write down the standard expression for the orthoprojection of the function $f \in L_p$ onto the space $S_{m-1}(\Delta_n)$ in terms of the elements of the orthonormal basis φ:

$$P_S(f, x) = \sum_{i=1}^{N} \varphi_i(x) \int_a^b \varphi_i(t) f(t)\, dt,$$

where $N = n + m - 1$. For $p \in [1, \infty]$, $1/p' = 1 - 1/p$ set

$$\|\cdot\|_l = \|\cdot\|_{L_p[t_l, t_{l+1}]}, \qquad \|\cdot\|_l' = \|\cdot\|_{L_{p'}[t_l, t_{l+1}]}.$$

Then, by virtue of Hölder and Minkowski inequalities with the help of (2) we obtain

$$\begin{aligned} \|P_S(f, \cdot)\|_k &\leqslant \sum_{i=1}^{N} \|\varphi_i\|_k \sum_{j=0}^{n-1} \|\varphi_i\|_j' \|f\|_j \\ &\leqslant c_m \sigma_m^\theta \sum_{i=1}^{N} \lambda^{|i-k|} \sum_{j=0}^{n-1} \lambda^{|i-j|} \|f\|_j \\ &= c_m \sigma_m^\theta \sum_{j=0}^{n-1} \|f\|_j \sum_{i=1}^{N} \lambda^{|i-k|} \lambda^{|i-j|} \\ &\leqslant c_{m,\varepsilon} \sigma_m^\theta \sum_{j=0}^{n-1} |k-j|\, \lambda^{|k-j|} \|f\|_j, \end{aligned}$$

i.e.,

$$\|P_S(f,\cdot)\|_k \leqslant c_{m,\varepsilon}\sigma_m^\theta \sum_{j=0}^{n-1} |k-j|\,\lambda^{|k-j|}\,\|f\|_j.$$

In the final estimate we use Young's inequality:

$$\begin{aligned}\|P_S(f,\cdot)\|_p &= \left\{\sum_{k=0}^{n-1} \|P_S(f,\cdot)\|_k^p\right\}^{1/p}\\ &\leqslant c_{m,\varepsilon}\sigma_m^\theta\left\{\sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1}|k-j|\,\lambda^{|k-j|}\,\|f\|_j\right)^p\right\}^{1/p}\\ &\leqslant c_{m,\varepsilon}\sigma_m^\theta\left\{\sum_{j=0}^{n-1}\|f\|_j^p\right\}^{1/p}\\ &\quad\times\left(\sup_j\sum_{k=0}^{n-1}|k-j|\,\lambda^{|k-j|}\right)^{1/p}\\ &\quad\times\left(\sup_k\sum_{j=0}^{n-1}|k-j|\,\lambda^{|k-j|}\right)^{1/p'}\\ &\leqslant c'_{m,\varepsilon}\sigma_m^\theta\,\|f\|_p.\end{aligned}$$

Theorem 1 is proved.

3. An Orthogonal Basis for $S_{m-1}(\Delta_n)$

We obtain a desired orthogonal spline basis as derivatives of appropriate fundamental splines; this idea goes back to J. H. Ahlberg and E. N. Nilson [1].

Complete the mesh Δ_n with the points $\{t_{-\nu}\}_{\nu=1}^{m-1}$ and $\{t_{n+\nu}\}_{\nu=1}^{m-1}$ which coincide with the endpoints of the interval $[a, b]$:

$$t_{-\nu} = t_0 = a, \qquad t_{n+\nu} = t_n = b, \quad \nu = \overline{1, m-1},$$

and denote the extension again by Δ_n.

Consider the family of splines $\Phi = \{\Phi_i\}_{i=1}^{n+m-1}$ of degree $2m-1$ from the set $S_{2m-1}(\Delta_n)$, defined by

$$\Phi_i = \arg\min_{g\in W_2^m}\{\|g^{(m)}\|_2 : g(t_j) = \delta_{ij},\, j = \overline{-m+1, i}\}.$$

Here, it is implied that

$$\begin{aligned}&\Phi_i^{(\mu)}(a) = 0, &&\mu = \overline{0, m-1}, \quad i = \overline{1, n+m-1};\\ &\Phi_{n+\nu}^{(\mu)}(b) = 0, &&\Phi_{n+\nu}^{(\nu)}(b) = 1, \quad \mu = \overline{0, \nu-1}, \quad \nu = \overline{1, m-1}.\end{aligned}$$

These are the fundamental splines on the widening meshes

$$\Delta_{ni} = \Delta_n \cap [t_{-m+1}, t_i],$$

which satisfy, besides the above interpolating conditions, the following boundary conditions

$$\Phi_i^{(m+\mu)}(t_i) = 0, \qquad \mu = \overline{0, m-2}, \qquad i = \overline{1, n};$$
$$\Phi_{n+\nu}^{(m+\mu)}(b) = 0, \qquad \mu = \overline{0, m-2-\nu}, \qquad \nu = \overline{1, m-2}.$$

Using the equalities

$$\Phi_i|_{\Delta_{nj}} = 0, \qquad i > j,$$

it is not hard to verify that the family Φ with number of elements $n+m-1$, which is equal to the dimension of the space $S_{m-1}(\Delta_n)$, turns out to be a system, orthogonal with respect to the inner product

$$(\Phi_i, \Phi_j) = \int_a^b \Phi_i^{(m)}(t)\, \Phi_j^{(m)}(t)\, dt.$$

Hence, it follows that the system $\varphi = \{\varphi_i\}_{i=1}^{n+m-1}$, consisting of the elements

$$\varphi_i := \Phi_i^{(m)} / \|\Phi_i^{(m)}\|_2,$$

is an orthonormal basis for $S_{m-1}(\Delta_n)$.

The classical basis for the space $S_{m-1}(\Delta_n)$ is the one $\{N_j\}_{j=-m+1}^{n-1}$ of B-splines with minimal supports:

$$\operatorname{supp} N_j = (t_j, t_{j+m}).$$

Since

$$\operatorname{supp} \varphi_i = (t_{-m+1}, t_i),$$

the system $\varphi = \{\varphi_i\}_{i=1}^{n+m-1}$ constructed is the result of the Gram–Schmidt orthogonalization process applied to the basis of B-splines.

4. Proof of Lemma 1 for $p \geq 2$

In this section we derive the estimate (2) for $p \in [2, \infty]$. The arguments used are based in turn on two statements which are proved in Section 6.

Lemma 2. *For all $p \in [2, \infty]$ and all $j < i$*

$$\|\varphi_i\|^2_{L_p[t_j, t_{j+1}]} \leq c_m \sum_{j'} \kappa_{j'}^{-1+2/p} \|\varphi_i\|^2_{L_2[t_{j'}, t_{j'+m}]},$$

where the sum is over all indices j' such that

$$[t_{j'}, t_{j'+m}] \supset [t_j, t_{j+1}], \qquad j'+m \leqslant i. \tag{3}$$

LEMMA 3. *There exists $\beta_1 = \beta_1(m)$, for which, for all ν such that $t_{\nu+1} \leqslant t_i$,*

$$\|\varphi_i\|^2_{L_2[0,\, t_\nu]} \leqslant \beta_1^{-1} (1 + \kappa_{\nu-m}^{-1} \kappa_{\nu-m+1})^{-1} \|\varphi_i\|^2_{L_2[t_{\nu-m+1},\, t_{\nu+1}]}.$$

The "one-sided" character of the statements presented is connected with special features of the splines φ_i: from the definition $\varphi_i(x) \equiv 0$ for $x \geqslant t_i$; thus the estimate (2) is to be proved only for indices $j < i$. Just the same statements (with corresponding changes in formulations) are valid for the mth derivatives of the usual "two-sided" fundamental splines.

Put

$$\eta := \eta_\varepsilon := \min\{\varepsilon, |1/2 - 1/p|\};$$

thus, in the case $p \geqslant 2$ we have

$$\theta = \theta_\varepsilon = (1/2 - \varepsilon - 1/p)_+,$$
$$\eta = \eta_\varepsilon = \min(\varepsilon, 1/2 - 1/p),$$
$$1/2 - 1/p = \theta + \eta$$

Now the estimate (2) for $p \geqslant 2$ is deduced in two steps.

(A) The case $j > i - 2m$, and hence $\min_{(3)} j' > i - 3m$.

Applying Lemma 3 as much as it is required, we have

$$\|\varphi_i\|^2_{L_2[0,\, t_{j'+m}]} \leqslant \beta_1^{-|i-j'-m|} (1 + \kappa_{j'}^{-1} \kappa_{i-m})^{-1} \|\varphi_i\|^2_{L_2[0,\, t_i]};$$

whence for all $0 \leqslant \varepsilon \leqslant \frac{1}{2}$ and $\lambda < 1$

$$\begin{aligned}
&\kappa_{j'}^{-1+2/p} \|\varphi_i\|^2_{L_2[t_{j'},\, t_{j'+m}]} \\
&\quad \leqslant \max(1, \beta_1^{-2m})\, \kappa_{j'}^{-2\theta} \kappa_{j'}^{-2\eta} (1 + \kappa_{j'}^{-1} \kappa_{i-m})^{-1} \|\varphi_i\|^2_{L_2[0,\, t_i]} \\
&\quad \leqslant \max(1, \beta_1^{-2m})\, \lambda^{-2m} \lambda^{|i-j|} \kappa_{\min}^{-2\theta} \kappa_{i-m}^{-2\eta} \|\varphi_i\|^2_{L_2[0,\, t_i]}
\end{aligned}$$

and to get the estimate (2) we have only to use Lemma 2.

(B) The case $j \leqslant i - 2m$, and hence $\max_{(3)} j' \leqslant i - 2m$.

From Lemma 3 it follows that there exists β,

$$\beta = \beta_m \leqslant \beta_1^m,$$

for which for all v, such that $t_{v+m} \leqslant t_i$, the inequality

$$\|\varphi_i\|^2_{L_2[0,\,t_v]} \leqslant \beta^{-1}(1+\kappa^{-1}_{v-m}\kappa_v)^{-1}\,\|\varphi_i\|^2_{L_2[t_v,\,t_{v+m}]} \tag{4}$$

is valid. With regard for the representation

$$\|\varphi_i\|^2_{L_2[0,\,t_v]} = \|\varphi_i\|^2_{L_2[0,\,t_{v-m}]} + \|\varphi_i\|^2_{L_2[t_{v-m},\,t_v]}$$

series of such inequalities with $v \in \{j' + \mu m\}^{\infty}_{\mu=0}$ implies the following estimate.

(b) For all j', such that $j' \leqslant i - 2m$,

$$\begin{aligned}\|\varphi_i\|^2_{L_2[t_{j'},\,t_{j'+m}]} &\leqslant \|\varphi_i\|^2_{L_2[0,\,t_{j'+m}]}\\ &\leqslant \left\{\prod_{\mu=0}^{k_{ij}} (1+\beta(1+\kappa^{-1}_{j'+\mu m}\kappa_{j'+(\mu+1)m}))\right\}^{-1}\\ &\quad\times \|\varphi_i\|^2_{L_2[0,\,t_{i'+m}]}.\end{aligned}$$

Here, $k_{ij} = (i' - j')/m$, and i' is such of indices from the sequence $\{j' + \mu m\}^{\infty}_{\mu=0}$ that

$$i - 2m < i' \leqslant i - m.$$

Hence, for the sake of brevity putting $\rho_i = \kappa_i \kappa^{-1}_{i+m}$, we have

$$\begin{aligned}&\kappa_{j'}^{-1+2/p}\,\|\varphi_i\|^2_{L_2[t_{j'},\,t_{j'+m}]}\\ &\quad\leqslant \kappa_{j'}^{-2\theta} \times \left\{\prod_{\mu=0}^{k_{ij}} (\rho^{2\eta}_{j'+\mu m}(1+\beta(1+\rho^{-1}_{j'+\mu m})))\right\}^{-1}\\ &\quad\times \kappa_{i'}^{-2\eta}\,\|\varphi_i\|^2_{L_2[0,\,t_{i'+m}]}.\end{aligned} \tag{5}$$

The first multiplier in the right-hand side is treated trivially

$$\kappa_{j'}^{-2\theta} \leqslant \kappa_{\min}^{-2\theta}.$$

Considering the second one, define for $p \geqslant 2$ the value ε^*_m as an upper bound for such $\varepsilon \geqslant 0$ which satisfy

$$\min_{\rho>0}\,((1+\beta)\,\rho^{2\varepsilon} + \beta\rho^{2\varepsilon-1}) \geqslant 1 + \delta_\varepsilon \tag{6}$$

with some $\delta_\varepsilon > 0$. The value of the minimum for $0 \leqslant 2\varepsilon \leqslant 1$ is equal to

$$[\beta/2\varepsilon]^{2\varepsilon}\,[(1+\beta)/(1-2\varepsilon)]^{1-2\varepsilon};$$

whence

$$\min(1, \beta) \leqslant 2\varepsilon^*_m \leqslant 1.$$

Hence for any $\varepsilon \in [0, \varepsilon_m^*)$ the majorant for the expression in the curly braces is just the quantity

$$\lambda_\varepsilon^{|i'-j'|} := (1+\delta_\varepsilon)^{-(1/m)|i'-j'|}.$$

An estimate for the last multiplier in (5) is obtained from (A):

$$\kappa_{i'}^{-2\eta} \|\varphi_i\|^2_{L_2[0, t_{i'+m}]} \leqslant \max(1, \beta_1^{-m})\, \lambda_\varepsilon^{-m} \lambda_\varepsilon^{|i-i'|} \kappa_{i-m}^{-2\eta} \|\varphi_i\|^2_{L_2[0, t_i]}.$$

Linking all estimates in one, from (5) with regard for the inequality $|j-j'|<m$, we find

$$\kappa_{j'}^{-1+2/p} \|\varphi_i\|^2_{L_2[0, t_{j'+m}]} \leqslant \max(1, \beta_1^{-m})\, \lambda_\varepsilon^{-2m} \lambda_\varepsilon^{|i-j|} \kappa_{\min}^{-2\theta} \kappa_{i-m}^{-2\eta} \|\varphi_i\|^2_{L_2[0, t_i]}$$

and for the final estimate (2) we refer to Lemma 2 once more. Thus, Lemma 1 for the case $p \geqslant 2$ is proved with $\varepsilon_m = \varepsilon_m^*$.

If we set in (5) $p=\infty$, $\varepsilon = \frac{1}{2}$, then we come to the relation

$$\|\varphi_i\|^2_{L_\infty[t_j, t_{j+1}]} \leqslant c_m \beta_m^{-|i-j|/m} \kappa_{i-m}^{-1} \|\varphi_i\|^2_{L_2[0, t_i]}, \tag{7}$$

which makes evident that $L_\infty^{(m)}$-norms of fundamental spline taken over the subintervals of arbitrary mesh Δ_n, if suitably normalized, are at least finite, and if the constant $\beta = \beta_m$, defined from inequality (4), satisfies the estimate $\beta > 1$, then such norms have exponential decay.

5. Proof of Lemma 1 for $p \leqslant 2$

To derive the estimate (2) for $p \leqslant 2$ we use just the same approach as in the previous case; however, the technical details differ somewhat. The corresponding auxiliary statements are the following.

LEMMA 2'. *For all $p \in [1, 2]$ and all $j < i-m$*

$$\|\varphi_i\|^2_{L_p[t_j, t_{j+1}]} \leqslant \kappa_j^{2/p-1} \left\{ \prod_{j'} \|\varphi_i\|^2_{L_2[t_{j'}, t_{j'+m}]} \right\}^{1/m},$$

where multiplication is over all indices j' such that

$$[t_{j'}, t_{j'+m}] \supset [t_j, t_{j+1}], \qquad j'+m \leqslant i. \tag{3'}$$

Unlike Lemma 2, this statement can be readily proved, since

$$\|\varphi_i\|^2_{L_p[t_j,\, t_{j+1}]} \leqslant h_j^{2/p-1} \|\varphi_i\|^2_{L_2[t_j,\, t_{j+1}]},$$

and we have only to majorize the right-hand side.

LEMMA 3′. *There exists $\beta_1 = \beta_1(m)$, for which, for all v such that $t_{v+m} \leqslant t_i$,*

$$\|\varphi_i\|^2_{L_2[0,\, t_v]} \leqslant \beta_1^{-1}(1 + \kappa_{v-1}\kappa_v^{-1})^{-1} \|\varphi_i\|^2_{L_2[t_v,\, t_{v+m}]}.$$

Let us proceed with the proof of (2) for $p \leqslant 2$. Now

$$\theta = \theta_\varepsilon = (1/p - 1/2 - \varepsilon)_+, \qquad \eta = \eta_\varepsilon = \min(\varepsilon,\, 1/p - 1/2);$$

therefore,

$$1/p - 1/2 = \theta + \eta.$$

(A_1') The case $j \geqslant i - m$.

The estimate (2) is evident: for all $0 \leqslant \varepsilon \leqslant \frac{1}{2}$ and $\lambda < 1$

$$\begin{aligned}\|\varphi_i\|^2_{L_p[t_j,\, t_{j+1}]} &\leqslant \kappa_{i-m}^{2/p-1} \|\varphi_i\|^2_{L_2[0,\, t_i]}\\ &\leqslant \lambda^{-m}\lambda^{|i-j|}\kappa_{\max}^{2\theta}\kappa_{i-m}^{2\eta} \|\varphi_i\|^2_{L_2[0,\, t_i]}.\end{aligned}$$

(A_2') The case $i - 2m \leqslant j < i - m$.

Divide the indices j' satisfying (3′) into two parts. For the first one write the trivial estimate

$$\|\varphi_i\|^2_{L_2[t_{j'},\, t_{j'+m}]} \leqslant \|\varphi_i\|^2_{L_2[0,\, t_i]}, \qquad i - 2m < j' \leqslant i - m,$$

and for the second—by virtue of Lemma 3′—the inequality

$$\|\varphi_i\|^2_{L_2[t_{j'},\, t_{j'+m}]} \leqslant \beta_1^{-1}(1 + \kappa_{j'+m-1}\kappa_{j'+m}^{-1})^{-1} \|\varphi_i\|^2_{L_2[0,\, t_i]},$$
$$i - 3m < j' \leqslant i - 2m.$$

Multiplying the left- and the right-hand sides of the above relations over the indices $j' = \overline{j - m + 1,\, j}$, we obtain

$$\begin{aligned}\prod_{j'} \|\varphi_i\|^2_{L_2[t_{j'},\, t_{j'+m}]} &\leqslant \|\varphi_i\|^{2m}_{L_2[0,\, t_i]} \prod_{\mu = j'+m}^{i-m} \beta_1^{-1}(1 + \kappa_{\mu-1}\kappa_\mu^{-1})^{-1}\\ &\leqslant \max(1,\, \beta_1^{-m})(1 + \kappa_{j-1}\kappa_{i-m}^{-1})^{-1} \|\varphi_i\|^{2m}_{L_2[0,\, t_i]}.\end{aligned}$$

Extracting the roots of mth degree and applying Lemma 2′, find that for all $0 \leqslant \varepsilon \leqslant 1/2m$ and $\lambda < 1$

$$\begin{aligned}\|\varphi_i\|^2_{L_p[t_j,\, t_{j+1}]} &\leqslant \max(1, \beta_1^{-1})\, \kappa_j^{2\theta} \kappa_j^{2\eta} (1 + \kappa_{j-1} \kappa_{i-m}^{-1})^{-1/m}\, \|\varphi_i\|^2_{L_2[0,\, t_i]} \\ &\leqslant \max(1, \beta_1^{-1})\, \lambda^{-2m} \lambda^{|i-j|} \kappa_{\max}^{2\theta} \kappa_{i-m}^{2\eta}\, \|\varphi_i\|^2_{L_2[0,\, t_i]}.\end{aligned}$$

(B′) The case $j < i - 2m$, and hence, $\max_{(3')} j' < i - 2m$.

For $1 \leqslant s \leqslant m$ put $j'_s := j - m + s$ and (like in Section 4) denote by i'_s an index from the sequence $\{j'_s + \mu m\}_{\mu=0}^{\infty}$ such that

$$i - 2m < i'_s \leqslant i - m.$$

With regard to the representation

$$\|\varphi_i\|^2_{L_2[0,\, t_\nu]} = \|\varphi_i\|^2_{L_2[0,\, t_{\nu-m}]} + \|\varphi_i\|^2_{L_2[t_{\nu-m},\, t_\nu]}$$

the reiterated use of Lemma 3′ gives

(b'_1) For all j'_s, such that $j'_s < i - 2m$,

$$\begin{aligned}\|\varphi_i\|^2_{L_2[t_{j'_s},\, t_{j'_s+m}]} \leqslant \|\varphi_i\|^2_{L_2[0,\, t_{j'_s+m}]} &\leqslant \left\{ \prod_{\mu=0}^{k_{ij}} (1 + \beta_1 (1 + \kappa_{j'_s+\mu m-1} \kappa_{j'_s+\mu m}^{-1})) \right\}^{-1} \\ &\quad \times \|\varphi_i\|^2_{L_2[0,\, t_{i'_s}]}.\end{aligned}$$

Multiplying the left- and the right-hand sides of these inequalities over $j'_s = \overline{j - m + 1, j}$, we have

(b'_2) For $j < i - 2m$ and j' such that $j - m < j' \leqslant j$,

$$\prod_{j'} \|\varphi_i\|^2_{L_2[t_{j'},\, t_{j'+m}]} \leqslant \left\{ \prod_{\nu=j}^{i-m} (1 + \beta_1 (1 + \kappa_{\nu-1} \kappa_\nu^{-1})) \right\}^{-1} \|\varphi_i\|^{2m}_{L_2[0,\, t_i]}.$$

Further, extract the root of mth degree and repeat the arguments from Section 4. Moreover, the limiting value ε_m^{**}, which implies for $\frac{1}{2} \leqslant 1/p < \frac{1}{2} + \varepsilon_m^{**}$ the exponential estimate of L_p-norms of φ_i, is defined as an upper bound of ε such that

$$\min_{\rho > 0} ((1 + \beta_1)\, \rho^{2m\varepsilon} + \beta_1 \rho^{2m\varepsilon - 1}) \geqslant 1 + \delta_\varepsilon, \qquad (6')$$

and it satisfies the inequality

$$\min(1, \beta_1) \leqslant 2m\varepsilon_m^{**} \leqslant 1.$$

At last we obtain that for $\varepsilon \in [0, \varepsilon_m^{**})$

$$\|\varphi_i\|^2_{L_p[t_j, t_{j+1}]} \leqslant \lambda_\varepsilon^{-m} \lambda_\varepsilon^{|i-j|} \kappa_{\max}^{2\theta} \kappa_{i-m}^{2\eta} \|\varphi_i\|^2_{L_2[0, t_i]},$$

which completes the proof of Lemma 1 for $p \leqslant 2$ with $\varepsilon_m = \varepsilon_m^{**}$.

Thus, we establish the estimate (2) with $\varepsilon_m = \min(\varepsilon_m^*, \varepsilon_m^{**})$, where the quantities ε_m^* and ε_m^* are define in relations (6) and (6′) by values of the constants β and β_1, which in turn appear in the statements of Lemmas 3 and 3′ and in inequality (4).

These relations, however, have principal limitations (due to the methods of the proof) in the following sense. If in (6) the estimate $\beta > 1$ is valid, then we obtain $\varepsilon_m^* = \frac{1}{2}$, and if we could get the same bound for the quantity ε_m^{**}, then de Boor's conjecture would be proved. But in (6′), whatever values the constant β_1 takes, we cannot exceed the limit $\varepsilon_m^{**} = \frac{1}{2m}$. In particular, this drawback does not allow us to get the estimate for the $L_1^{(m)}$-norm similar to (7); therefore, to prove Theorem 2 we use other methods.

6. Proof of Main Inequalities

6.1. *Auxiliary Statements*

Set

$$h_{\nu, r} = t_{\nu + r} - t_\nu, \qquad h_\nu = h_{\nu, 1}.$$

Further, for the elements of the basis $\{N_l\}_{l=-m+1}^{n-1}$ of normalized B-splines which satisfy the conditions

$$\operatorname{supp} N_l = (t_l, t_{l+m}), \qquad \sum N_l \equiv 1,$$

define the splines

$$N_{lp}(\cdot) = m^{1/p} \kappa_l^{-1/p} N_l(\cdot), \qquad p \in [1, \infty]. \tag{8}$$

While proving Lemmas 2, 3, 3′ we rely upon the following statements.

Lemma A. *For any vector $a = (a_l)_{l=-m+1}^{n-1}$ for $p \in [1, \infty]$*

$$\begin{aligned} D_m^{-1} \|a\|_{l_p} &\leqslant \left\| \sum a_l N_{lp} \right\|_{L_p[0,1]} \leqslant \|a\|_{l_p}, \\ D_m^{-1} m^{1/p} |a_\nu| &\leqslant \left\| \sum a_l N_{lp} \right\|_{L_p[t_\nu, t_{\nu+m}]}. \end{aligned} \tag{9}$$

LEMMA B. *If* $g \in W_2^m[0, 1]$ *and* $g(t_{\nu+\mu}) = 0$, $\mu = \overline{0, m-1}$, *then*

$$\|g^{(m-1-k)}\|_{L_\infty[t_\nu, t_{\nu+m-1}]} \leqslant c_{1,m,k} h_{\nu,m-1}^{k+1/2} \|g^{(m)}\|_{L_2[t_\nu, t_{\nu+m-1}]},$$
$$k = \overline{0, m-1}.$$

LEMMA C. *If* $g \in \pi_{2m-1}$ *(i.e., is an algebraic polynomial of degree* $2m-1$*), then*

$$\|g^{(m+k)}\|_{L_\infty[t_\nu, t_{\nu+1}]} \leqslant c_{2,m,k} h_\nu^{-(k+1/2)} \|g^{(m)}\|_{L_2[t_\nu, t_{\nu+1}]},$$
$$k = \overline{0, m-1}.$$

The last auxiliary statement is concerned with the fundamental splines $\{\Phi_i\}_{i=1}^{n+m-1}$, which determine the orthonormal system $\varphi = \{\varphi_i\}_{i=1}^{n+m-1}$ by the rule $\varphi_i = \Phi_i^{(m)}/\|\Phi_i^{(m)}\|_2$.

LEMMA D. *There exists* $\beta_0 = \beta_0(m)$ *for which, for all* ν *such that* $t_{\nu+m-1} < t_i$,

$$\|\Phi_i^{(m)}\|_{L_2[0, t_\nu]}^2 \leqslant \beta_0^{-1} \|\Phi_i^{(m)}\|_{L_2[t_\nu, t_{\nu+m-1}]}^2.$$

Lemma A is due to C. de Boor [6]. Lemma B, if we do not care for exact constants, is elementarily proved by virtue of the Rolle theorem. Lemma C is a Markov-type inequality for different metrics and when exact constants are not required uses nothing more than finite dimensionality. Lemma D is what the "exponential decay" property of a fundamental spline is based on and is proved, e.g., in [9].

6.2. *Proof of Lemma* 2.

Expand the spline φ_i in the basis $\{N_{lp}\}$ for $p = 2$:

$$\varphi_i = \sum_{l=m+1}^{n-1} b_l N_{l2}.$$

For a given j define the spline ψ_j by

$$\psi_j = \sum_{l=j-m+1}^{j} b_l N_{l2}.$$

Then $\psi_j \equiv \varphi_i$ in the interval $[t_j, t_{j+1}]$ and application of Lemma A for $p \geqslant 2$ gives

$$\begin{aligned}\|\varphi_i\|^2_{L_p[t_j,\, t_{j+1}]} &= \|\psi_j\|^2_{L_p[t_j,\, t_{j+1}]}\\ &\leqslant c_m \left\{ \sum_{l=j-m+1}^{j} (\kappa_l^{-1+2/p} b_l^2)^{p/2} \right\}^{2/p}\\ &\leqslant c_m \sum_{l=j-m+1}^{j} \kappa_l^{-1+2/p} b_l^2\\ &\leqslant c'_m \sum_{l=j-m+1}^{j} \kappa_l^{-1+2/p} \|\varphi_i\|^2_{L_2[t_l,\, t_{l+m}]},\end{aligned}$$

which is required.

6.3. *Proof of Lemma* 3.

Since

$$\varphi_i = \Phi_i^{(m)} / \|\Phi_i^{(m)}\|_2,$$

it suffices to show that if v satisfies the condition $t_{v+1} \leqslant t_i$, then

$$\|\Phi_i^{(m)}\|^2_{L_2[0,\, t_v]} \leqslant \beta_1^{-1} (1 + \kappa_{v-m}^{-1} \kappa_{v-m+1})^{-1} \|\Phi_i^{(m)}\|^2_{L_2[t_{v-m+1},\, t_{v+1}]}. \tag{10}$$

Consider two possible variants of correlation between the parts $[t_v, t_{v+1}]$ and $[t_{v-m}, t_v]$ of the interval $[t_{v-m}, t_{v+1}]$.

(i) $h_v \leqslant \kappa_{v-m}$.

Then,

$$\kappa_{v-m} \geqslant \tfrac{1}{2}(\kappa_{v-m} + h_v) = \tfrac{1}{2}(h_{v-m} + \kappa_{v-m+1}) \geqslant \tfrac{1}{2}\kappa_{v-m+1},$$

i.e.,

$$\kappa_{v-m}^{-1} \kappa_{v-m+1} \leqslant 2.$$

Combining this estimate with Lemma C, derive that

$$\begin{aligned}\|\Phi_i^{(m)}\|^2_{L_2[0,\, t_v]} &\leqslant (1 + \beta_0^{-1}) \|\Phi_i^{(m)}\|^2_{L_2[t_{v-m+1},\, t_v]}\\ &\leqslant (1 + \beta_0^{-1}) \|\Phi_i^{(m)}\|^2_{L_2[t_{v-m+1},\, t_{v+1}]}\\ &\leqslant 3(1 + \beta_0^{-1})(1 + \kappa_{v-m}^{-1} \kappa_{v-m+1})^{-1} \|\Phi_i^{(m)}\|^2_{L_2[t_{v-m+1},\, t_{v+1}]}.\end{aligned}$$

Thus, in case (i) estimate (10) holds with

$$\beta_1 = \tfrac{1}{3}(1 + \beta_0^{-1})^{-1}. \tag{11}$$

(ii) $h_v \geqslant \kappa_{v-m} = h_{v-m} + h_{v-m+1,\, m-1}$.

The fundamental spline Φ_i has a piecewise polynomial structure and $\Delta_{n,i-1}$-mesh of zeroes. So, integrate the quantity $\|\Phi_i^{(m)}\|^2_{L_2[0,t_\nu]}$ by parts and apply Lemmas B and C:

$$\begin{aligned}\|\Phi_i^{(m)}\|^2_{L_2[0,t_\nu]} &= \sum_{k=0}^{m-1} (-1)^k \Phi_i^{(m-1-k)}(t_\nu)\, \Phi_i^{(m+k)}(t_\nu)\\ &\leqslant \sum_{k=0}^{m-1} \|\Phi_i^{(m-1-k)}\|_{L_\infty[t_{\nu-m+1},t_\nu]}\\ &\quad\times \|\Phi_i^{(m+k)}\|_{L_\infty[t_\nu,t_{\nu+1}]}\\ &\leqslant \sum_{k=0}^{m-1} c_{1,m,k} c_{2,m,k} (h_{\nu-m+1,m-1} h_\nu^{-1})^{k+1/2}\\ &\quad\times \|\Phi_i^{(m)}\|_{L_2[t_{\nu-m+1},t_\nu]} \|\Phi_i^{(m)}\|_{L_2[t_\nu,t_{\nu+1}]}\\ &\leqslant 2^{1/2} c_{3,m} \kappa_{\nu-m}^{1/2} \kappa_{\nu-m+1}^{-1/2}\\ &\quad\times \|\Phi_i^{(m)}\|_{L_2[0,t_\nu]} \|\Phi_i^{(m)}\|_{L_2[t_{\nu-m+1},t_{\nu+1}]}.\end{aligned}$$

In the final inequality of this series we had put

$$c_{3,m} = \sum_{k=0}^{m-1} c_{1,m,k} c_{2,m,k}$$

and had used the relations

$$h_\nu \geqslant h_{\nu-m+1,m-1}, \qquad \kappa_{\nu-m} \geqslant h_{\nu-m+1,m-1},$$
$$h_\nu \geqslant \tfrac{1}{2}(h_{\nu-m+1,m-1} + h_\nu) = \tfrac{1}{2}\kappa_{\nu-m+1},$$

which followed from (ii).

Thus, we obtain

$$\begin{aligned}\|\Phi_i^{(m)}\|^2_{L_2[0,t_\nu]} &\leqslant 2c_{3,m}^2 \kappa_{\nu-m} \kappa_{\nu-m+1}^{-1} \|\Phi_i^{(m)}\|^2_{L_2[t_{\nu-m+1},t_{\nu+1}]}\\ &\leqslant 4c_{3,m}^2 (1 + \kappa_{\nu-m}^{-1}\kappa_{\nu-m+1})^{-1} \|\Phi_i^{(m)}\|^2_{L_2[t_{\nu-m+1},t_{\nu+1}]},\end{aligned}$$

i.e., the estimate (10) with such value for β_1:

$$\beta_1 = \tfrac{1}{4}\left(\sum_{k=0}^{m-1} c_{1,m,k} c_{2,m,k}\right)^{-2}. \tag{12}$$

6.3. *Proof of Lemma* 3′.

We must show that under the condition $t_{\nu+m} \leqslant t_i$

$$\|\Phi_i^{(m)}\|^2_{L_2[0,t_\nu]} \leqslant \beta_1^{-1}(1 + \kappa_{\nu-1}\kappa_1^{-1})^{-1} \|\Phi_i^{(m)}\|^2_{L_2[t_\nu,t_{\nu+m}]}.$$

Repeat word for word the arguments of Section 6.3 accurate up to the symmetry with respect to the point t_ν. Consider the interval $[t_{\nu-1}, t_{\nu+m}]$ and two variants of correlation between its two parts $[t_{\nu-1}, t_\nu]$ and $[t_\nu, t_{\nu+m}]$:

(i′) $h_{\nu-1} \leqslant \kappa_\nu$;

(ii′) $h_{\nu-1} \geqslant \kappa_\nu = h_{\nu, m-1} + h_{\nu+m-1}$.

Theorem 1 is completely proved.

7. Comments

The fact that the L_2-norm of mth derivative of fundamental spline decays exponentially for arbitrary mesh Δ_n (briefly, L_2-property) was discovered by C. de Boor [7] and it turned out to be very useful in spline-interpolation problems [7, 9, 10]. An elegant proof of such a property within the variational spline theory is due to Yu. N. Subbotin [10]. Some omissions in his arguments were corrected in [9].

As became recently known to us, the idea to estimate the $L_2^{(m)}$-norms of a fundamental spline using integration by parts coupled with Lemmas B and C has been offered earlier by Yu. N. Subbotin as one more method for proving the L_2-property (published in the doctoral thesis of his student [2]).

Our approach to C. de Boor's problem described in Section 2 implies that the fundamental spline satisfies the L_p-property for $p = 1$ and ∞. Now a spline has piecewise polynomial structure and L_p-norms of polynomials are equivalent in a fixed interval. Thus, we conclude that the rate of exponential decay of the $L_2^{(m)}$-norm of a fundamental spline must depend on the rate of a nonuniformity of the mesh Δ_n, and we ought to attain the estimate

$$\|\Phi_i^{(m)}\|^2_{L_2[0, t_\nu]} \leqslant \beta_*^{-1}(\kappa_{\nu-m}^{-1}\kappa_\nu + \kappa_{\nu-m}\kappa_\nu^{-1})^{-1} \|\Phi_i^{(m)}\|^2_{L_2[t_\nu, t_{\nu+m}]}$$

with a constant $\beta_* > 1$. The possibility of such an inequality by order is established by Lemmas 3 and 3′; in the last one, however, we fail to obtain the exponent required.

The value of β_1, which in our method determines the radius of L_p-norms, for which the quantity $l_{m-1}(\Delta_n)_p$ is unconditionally bounded, could be practically computed on the basis of the estimates (11)–(12), but they are certainly quite rough. Theoretically, it is possible to compute the exact key constants such as β_1 (in particular β_*) as eigenvalues of some special matrices of order $(2m-2) \times (2m-2)$, but in practice, in view of cumbersomeness of matrices involved, we fail to go further than the investigated cases $m = 2, 3$.

8. PROOF OF THEOREM 2

Define the matrix

$$A_p = A_{p,m-1}(\Delta_n) = \left\{ \int_a^b N_{ip}(t)\, N_{jp'}(t)\, dt \right\}_{i,j=-m+1}^{n-1}$$

or order $N \times N$, where $N = n + m - 1$. It consists of all possible inner product $(N_{ip}, N_{jp'})$ of p- and p'-normalized B-splines of degree $m-1$ on mesh Δ_n, which were introduced in (8).

C. de Boor [3] proved that

$$l_{m-1}(\Delta_n)_p \leqslant \|A_p^{-1}\|_{l_p \to l_p}.$$

Here we give a direct estimate of the norm of the inverse matrix A_p^{-1} for $p = \infty$, and this leads to Theorem 2. For this purpose we need two lemmas.

LEMMA 4. *For each $M \in \mathbb{N}$ and any functions $\{f_i\}_{i=1}^M$ and $\{g_i\}_{i=1}^M$,*

$$\det\{(f_i, g_j)\}_1^M = (M!)^{-1} \int_{I^M} \det\{f_i(z_k)\}_1^M \det\{g_j(z_k)\}_1^M\, dz,$$

where I^M is an M-dimensional cube $[a,b]^M$, and $dz = dz_1 \cdots dz_M$.

LEMMA 5. *For any $m, n \in \mathbb{N}$, and $L \leqslant N = n + m - 1$, and $p \in [1, \infty]$*

$$D_m^{-L} \leqslant (L!)^{-1/p} \|\det\{N_{i_s,p}(z_{k_t})\}_{s,t=1}^L\|_{L_p(I^L)} \leqslant 1.$$

Lemma 4 is due to G. Polya and G. Szego [8, Vol. 1, part 2, Problem 68] and can be proved by induction on M. Lemma 5 is derived by induction on L combined with the estimate (9) of Lemma A.

Let us now evaluate the elements of the matrix $A_\infty^{-1} = \{a_{ij}^{(-1)}\}_1^N$, by the well-known formula

$$a_{ij}^{(-1)} = (\det A_\infty)^{-1} A_{ji},$$

where A_{ji} is the algebraic adjoint of an element a_{ji} of the matrix A_∞ in the determinant $\det A_\infty$.

It is not hard to see that for all $p \in [1, \infty]$

$$\det A_p = \det A_2.$$

Applying Lemmas 4 and 5, we have

$$\begin{aligned}\det A_\infty &= \det A_2 = \det\{(N_{i2}, N_{j2})\}_1^N \\ &= (N!)^{-1} \int_{I^N} \det\{N_{i2}(z_k)\} \det\{N_{j2}(z_k)\}\, dz \\ &= (N!)^{-1} \|\det\{N_{i2}(z_k)\}\|_{L_2(I^N)}^2 \geqslant D_m^{-2N}.\end{aligned}$$

Similarly, for any $1 \leqslant i, j \leqslant N$

$$\begin{aligned}
|A_{ji}| &= |\det\{(N_{\nu\infty}, N_{\mu 1})\}_{1,\, \nu \neq j,\, \mu \neq i}^{N}| \\
&= (N-1)! \left| \int_{I^{N-1}} \det\{N_{\nu\infty}(z_k)\} \det\{N_{\mu 1}(z_k)\}\, dz \right| \\
&\leqslant (N-1)!^{-1} \|\det\{N_{\mu 1}(z_k)\}\|_{L_1(I^{N-1})} \\
&\quad \times \|\det\{N_{\nu\infty}(z_k)\}\|_{L_\infty(I^{N-1})} \leqslant 1.
\end{aligned}$$

Thus, for any $1 \leqslant i, j \leqslant N$

$$|a_{ij}^{(-1)}| \leqslant D_m^{2N},$$

$$\|A_\infty^{-1}\|_{l_\infty \to l_\infty} = \sup_i \sum_{j=1}^{N} |a_{ij}^{(-1)}| \leqslant N D_m^{2N},$$

and therefore,

$$l_{m-1}(\Delta_n)_\infty \leqslant N D_m^{2N},$$

where $N = n + m - 1$, and D_m is the constant from inequality (9) of Lemma A. Theorem 2 is proved.

Remark. One of the referees has pointed out that he had presented such a result (with an alternative proof) at Columbia at one of the SouthEast Approximation Theory conferences, but he has never published it.

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